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Uniform laws of the iterated logarithm for Lipschitz classes of functions

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1. Introduction

Let $\{[0, 1], \mathscr{F}, P\}$ be the unit interval with Lebesgue measurability and Lebesgue measure P. For $1/2<\alpha$, let Λ_α be the class of real-valued functions f on $[0, 1]$ with $f(0)=f(1)$, $\int_0^1 f(x)\,dx=0$ and satisfying a Lipschitz condition

$$|f(x)-f(y)| \leqq |x-y|^\alpha, \quad 0 \leqq x, y \leqq 1.$$

Extend the functions of Λ_α with period 1.

The purpose of this paper is to prove the following two theorems.

Theorem 1. *Let $\{n_k, k\geqq 1\}$ be a sequence of real numbers satisfying*

$$n_{k+1}/n_k \geqq 1+c/k^\delta \quad (c>0) \tag{1.1}$$

for some $0<\delta<1/2$. Then

$$\limsup_{N\to\infty} \sup_{f\in\Lambda_\alpha} \Big|\sum_{k\leqq N} f(n_k x)\Big|/(N\log\log N)^{1/2} \leqq C \quad a.s.$$

(with respect to the Lebesgue measure on $[0, 1]$). The constant C depends only on α and δ.

We say that a sequence $\{n_k\}$ of integers satisfies condition B_2 if there is a constant C such that the number of solutions of the equation $n_k \pm n_l = v$ does not exceed C for any $v \geqq 0$.

Theorem 2. *Let $\{n_k, k\geqq 1\}$ be a sequence of integers satisfying condition B_2 and*

$$n_{k+1}/n_k \geqq 1+c/k^\delta \quad (c>0) \tag{1.2}$$

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with $1/2 \leqq \delta \leqq 1$. *Then for each* α *with* $1/2+\delta/2<\alpha$,

$$\limsup_{N\to\infty} \sup_{f\in A_\alpha} \Big|\sum_{k\leqq N} f(n_k x)\Big|/(N\log\log N)^{1/2} \leqq C$$

for almost all $x\in[0, 1]$, *where* C *is a constant depending on* α, δ *and the constant in the* B_2 *condition of the sequence* $\{n_k, k\geqq 1\}$.

The results in Theorem 1 improve upon Theorem 3.2 of KAUFMAN and PHILIPP [10] who, instead of (1.1), assume the more restrictive condition

$$n_{k+1}/n_k \geqq q > 1.$$

2. Proof of Theorem 1

In the course of the proof of Theorem 1 we shall prove the following two propositions.

Proposition 1. *Let* $\{n_k, k\geqq 1\}$ *be as in Theorem* 1. *Then there exist positive constants* A *and* C_1 *such that*

$$P\Big\{\Big|\sum_{k=1}^{N} \exp(2\pi i n_k x)\Big| \geqq AR(N\log\log N)^{1/2}\Big\} \leqq C_1 \exp(-10R\log\log N)$$

for all $R\geqq 1$, $N\geqq 1$.

Proposition 2. *Let* $\{n_k, k\geqq 1\}$ *be as in Theorem* 1. *Then*

$$P\Big\{\max_{1\leqq m<N}\Big|\sum_{k=N+1}^{N+m} \exp(2\pi i n_k x)\Big| \geqq AR(N\log\log N)^{1/2}\Big\} \leqq C_1 \exp(-10R\log\log N)$$

for all $R\geqq 1$, $N\geqq 1$. *Here* A *and* C_1 *are as in Proposition* 1.

Before we prove these propositions we shall deduce Theorem 1 from them. We need to introduce some notations. For integers h, N, $H\geqq 1$, we put

$$F(H, N, h) = \Big|\sum_{k=H+1}^{H+N} \exp(2\pi i h n_k x)\Big|. \tag{2.1}$$

Then

$$F(0, N, h) \leqq F(0, 2^n, h) + \max_{1\leqq m<2^n} F(2^n, m, h) \tag{2.2}$$

where $n=[\log N/\log 2]$. Here $[x]$ denotes the largest integer not exceeding x for any real number x. Put

$$\varphi(N) = (N\log\log N)^{1/2} \tag{2.3}$$

and define the events (here and throughout $\log^+ x = \log(\max(e, x))$)

$$G(n, h) = \{F(0, 2^n, h) \geqq A \log^+ |h| \varphi(2^n)\}; \quad G_n = \bigcup_{|h| \geqq 1} G(n, h);$$

$$H(n, h) = \{\max_{1 \leqq m < 2^n} F(2^n, m, h) \geqq A \log^+ |h| \varphi(2^n)\}; \quad H_n = \bigcup_{|h| \geqq 1} H(n, h),$$

where A is the constant appearing in Proposition 1.

We now show that with probability 1 only finitely many G_n and H_n occur. In fact, by Proposition 1,

$$P\{G(n, h)\} \ll \exp(-2 \log^+ |h| \log n).$$

Thus

$$P(G_n) \ll \sum_{|h| \geqq 1} \exp(-2 \log^+ |h| \log n) \ll n^{-2}.$$

Also, by Proposition 2,

$$P\{H(n, h)\} \ll \exp(-2 \log^+ |h| \log n).$$

Thus

$$P(H_n) \ll \sum_{|h| \geqq 1} \exp(-2 \log^+ |h| \log n) \ll n^{-2}.$$

Now we can conclude that

$$F(0, N, h) \ll \log^+ |h| \varphi(N) \quad \text{a.s.} \tag{2.4}$$

for all $|h| \geqq 1$ by (2.2).

In [10], Kaufman and Philipp showed that if $f \in \Lambda_\alpha$ $(\alpha > 1/2)$, then the coefficients a_h of the Fourier series of f

$$f(x) = \sum_{|h| \geqq 1} a_h \exp(2\pi i h x) \tag{2.5}$$

satisfies

$$\sum_{|h| \geqq N} a_h \exp(2\pi i h x) \ll N^{-1/2} \tag{2.6}$$

uniformly in x, and

$$\sum_{|h| \geqq 1} |a_h|^2 |h| (\log^+ |h|)^4 \ll 1. \tag{2.7}$$

In fact, (2.7) can be replaced by

$$\sum_{|h| \geqq 1} |a_h|^2 |h|^{1+\varepsilon} \ll 1 \tag{2.8}$$

for any ε with $0 < \varepsilon < 2\alpha - 1$ since in [15], formula (3) on page 136, we have

$$\sum_{h=2^{v-1}-1}^{2^v} |a_h|^2 < 2C2^{-2\alpha v}$$

for an absolute constant C. Thus

(2.9) $$\sum_{h=2^{v-1}-1}^{2^v} |a_h|^2 h^{1-\varepsilon} \leqq 2^{(1+\varepsilon)v} \sum_{h=2^{v-1}-1}^{2^v} |a_h|^2 \leqq 2C2^{-(2\alpha-1-\varepsilon)v}.$$

Since $\varepsilon<2\alpha-1$, (2.8) follows from (2.9).

Now for each $f\in\Lambda_\alpha$ $(\alpha>1/2)$, we have if $0<\varepsilon'<\varepsilon<2\alpha-1$

(2.10)
$$\Big|\sum_{k\leqq N} f(n_k x)\Big| \leqq \Big|\sum_{1\leqq|h|\leqq N/2} a_h \sum_{k\leqq N} \exp(2\pi i n_k hx)\Big| + \Big|\sum_{k\leqq N}\sum_{|h|>N/2} a_h \exp(2\pi i n_k hx)\Big| \ll$$
$$\ll \Big(\sum_{1\leqq|h|\leqq N/2} |a_h|^2 |h|^{1+\varepsilon}\Big)^{1/2}\Big(\sum_{1\leqq|h|\leqq N/2} |h|^{-1-\varepsilon} F^2(0, N, h)\Big)^{1/2} + NN^{-1/2} \ll$$
$$\ll 1\Big(\sum_{1\leqq|h|\leqq N/2} |h|^{-1-\varepsilon}|h|^{\varepsilon'}\Big)^{1/2}\varphi(N) + N^{1/2} \ll \varphi(N)$$

by (2.5), (2.1), (2.6), (2.8), and (2.4).

2.1 Proof of Proposition 1. (The proof is based on the idea of the proof of Proposition 4.2.1 of PHILIPP [11].)

Lemma 1. *For* $1\leqq j<k$ *we have* $n_j/n_k \leqq 2^{-c(k-j)/k^\delta}$.

The proof is very simple (cf. [1], p. 211).

Choose ε so that

(2.11) $$\delta < \varepsilon/(1+\varepsilon) < 1/2.$$

We now divide $\mathbf{Z}^+$ into blocks (without gaps) such that

$$H_1 < I_1 < H_2 < I_2 < \ldots < H_j < I_j < \ldots \quad \text{(say)},$$
$$\operatorname{card}(H_j) = \operatorname{card}(I_j) = [j^\varepsilon].$$

Let

(2.12) c_j be the smallest element of H_j, and let d_j be the largest element of H_j.

For $v\in\mathbf{Z}^+$, we write

(2.13) $$m_v = [\varrho_v + 5\log v]$$

where $2^{\varrho_v} \leqq n_v < 2^{\varrho_v+1}$. Put

(2.14) $$\psi_v(x) = \cos 2\pi n_v(k/2^{m_v}) \quad \text{if} \quad x\in[k2^{-m_v}, (k+1)2^{-m_v})$$
$$(k = 0, 1, \ldots, 2^{m_v}-1).$$

Now write

(2.15) $$T_j = \sum_{v\in H_j} \cos 2\pi n_v x, \quad D_j = \sum_{v\in H_j} \psi_v(x), \quad \bar{D}_j = D_j - E(D_j|D_1, \ldots, D_{j-1}).$$

(The definition of D_j was introduced by BERKES [1].)

Lemma 2. $\|T_j - D_j\|_\infty \ll C_j^{-3}$.

Proof. We have

$$|\cos 2\pi n_v x - \psi_v(x)| \leqq 2\pi n_v 2^{-m_v} \ll n_v 2^{-\varrho_v} v^{-4} \ll v^{-4}$$

by (2.13). Thus

$$\|T_j - D_j\|_\infty \ll \sum_{v \in H_j} v^{-4} \ll C_j^{-3}.$$

by (2.12).

Lemma 3. $E(T_j | D_1, \ldots, D_{j-1}) \ll j^{-2}$ *a.s.*

Proof. Let

$$A = [k2^{-m_{d_{j-1}}}, (k+1)2^{-m_{d_{j-1}}})$$

for any $k = 0, 1, \ldots, 2^{m_{d_{j-1}}}$. Then, putting $m'_v = 2\pi n_v / 2^{m_{d_{j-1}}}$, we have

$$P^{-1}(A) \int_A T_j = \int_k^{k+1} \sum_{v \in H_j} \cos m'_v x \ll$$

$$\ll \sum_{v \in H_j} 1/m'_v \ll j^\varepsilon / m'_{c_j} \ll j^\varepsilon 2^{m_{d_{j-1}}} / n_{c_j} \ll$$

$$\ll j^\varepsilon d_{j-1}^5 (n_{d_{j-1}} / n_{c_j}) \ll j^\varepsilon d_{j-1}^5 2^{-cj^\varepsilon / j^{(1+\varepsilon)\delta}} \ll j^{\varepsilon + 5(1+\varepsilon)} 2^{-cj^{\varepsilon - (1+\varepsilon)\delta}} \ll j^{-2}$$

by Lemma 1 and (2.12).

Lemma 4. $E(T_j^2 | D_1, \ldots, D_{j-1}) \ll j^\varepsilon$ *a.s.*

Proof. Let A be as in the proof of Lemma 3.

$$P^{-1}(A) \int_A T^2 = \int_k^{k+1} \Big(\sum_{v \in H_j} \cos m'_v x\Big)^2 \ll$$

$$\ll j^\varepsilon + \Big|\int_k^{k+1} \sum_{\mu < v \in H_j} \cos m'_\mu x \cos m'_v x\Big| + \Big|\int_k^{k+1} \sum_{v \in H_j} \cos 2m'_v x\Big| \ll$$

$$\ll j^\varepsilon + j^\varepsilon / m'_{c_j} + j^\varepsilon \sum_{v \in H_j} 1/(m'_{v+1} - m'_v) \ll j^\varepsilon + j^\varepsilon d_j^\delta \sum_{v \in H_j} 1/m'_v \ll$$

$$\ll j^\varepsilon + j^{\varepsilon + (1+\varepsilon)\delta} j^{\varepsilon + 5(1+\varepsilon)} 2^{-cj^{\varepsilon - (1+\varepsilon)\delta}} \ll j^\varepsilon \quad \text{a.s.}$$

by (2.13) and (2.12).

Lemma 5. (1) $E(D_j | D_1, \ldots, D_{j-1}) \ll j^{-2}$ *a.s.*
(2) $E(\bar{D}_j^2 | D_1, \ldots, D_{j-1}) \ll j^\varepsilon$ *a.s.*

Proof. (1) follows from Lemma 2 and (2.13). (2) follows from Lemma 4 and the following computation: Since $\operatorname{Var}(X|\mathscr{A}) \leqq E(X^2|\mathscr{A})$, we have, by (2.15),

$$E(\bar{D}_j^2|D_1, \ldots, D_{j-1}) \leqq E(D_j^2|D_1, \ldots, D_{j-1}) + E^2(D_j|D_1, \ldots, D_{j-1}) \ll$$
$$\ll E(|D_j^2 - T_j^2|\,|D_1, \ldots, D_{j-1}) + E(T_j^2|D_1, \ldots, D_{j-1}) \ll$$
$$\ll j^\varepsilon \|T_j - D_j\|_\infty + E(T_j^2|D_1, \ldots, D_{j-1}) \ll j^{\varepsilon - 3(1+\varepsilon)} + E(T_j^2|D_1, \ldots, D_{j-1}).$$

Lemma 6. *Let $B(\geqq 1)$ be the constant implied by $\ll$ in Lemma 5(2). Then we have as $M \to \infty$*

$$P\Big\{\Big|\sum_{j \leqq M} \bar{D}_j\Big| > 16RB(M^{\varepsilon+1} \log\log M^{\varepsilon+1})^{1/2}\Big\} \ll \exp(-12RB \log\log M^{\varepsilon+1}).$$

Proof. Put

$$u_n = \sum_{j \leqq n} \bar{D}_j, \quad n \leqq M,$$
$$= u_M, \quad n > M;$$
$$s_n^2 = \sum_{j \leqq n} E(\bar{D}_j^2|D_1, \ldots, D_{j-1}), \quad n \leqq M,$$
$$= s_M^2, \qquad n > M;$$
$$c = M^\varepsilon, \quad \lambda = 2(\log\log M^{\varepsilon+1})^{1/2} M^{-(\varepsilon+1)/2}, \quad K = 4RBM^{\varepsilon+1};$$
$$T_n = \exp\big(\lambda u_n - (1/2)\lambda^2(1 + (1/2)\lambda c)s_n^2\big).$$

Thus as in the proof of Lemma 4.2.9 of PHILIPP [11],

$$P\Big\{\sum_{j \leqq M} \bar{D}_j > 8RB(M^{\varepsilon+1} \log\log M^{\varepsilon+1})^{1/2}\Big\} \leqq P\big\{\sup_{n \geqq 0} u_n > \lambda K\big\} \leqq$$
$$\leqq P\big\{\sup_{n \geqq 0} T_n > \exp(\lambda^2 K - \lambda^2 BM^{\varepsilon+1})\big\} \leqq \exp(\lambda^2 BM^{\varepsilon+1} - \lambda^2 K) \leqq$$
$$\leqq \exp(-12RB \log\log M^{\varepsilon+1}).$$

Lemma 7. *There is a positive constant A_1 such that*

$$P\Big\{\Big|\sum_{j \leqq M} T_j\Big| \geqq A_1 R(M^{\varepsilon+1} \log\log M^{\varepsilon+1})^{1/2}\Big\} \ll \exp(-12R \log\log M^{\varepsilon+1}).$$

Proof. This lemma is a consequence of Lemmas 2, 5(1) and 6 together with the following equality:

$$\sum_{j \leqq M} T_j = \sum_{j \leqq M} (T_j - D_j) + \sum_{j \leqq M} (D_j - \bar{D}_j) + \sum_{j \leqq M} \bar{D}_j.$$

Similarly, we can prove that

$$P\Big\{\Big|\sum_{j \leqq M} T_j'\Big| \geqq A_1 R(M^{\varepsilon+1} \log\log M^{\varepsilon+1})^{1/2}\Big\} \ll \exp(-12R \log\log M^{\varepsilon+1})$$

where

$$T_j' = \sum_{v\in I_j} \cos 2\pi n_v x. \tag{2.16}$$

Hence if $N\in H_{M+1}$, then, by (2.15), (2.16) and Lemma 7,

$$P\Big\{\Big|\sum_{k\le N} \cos 2\pi n_k x\Big| \ge 3A_1 R(N\log\log N)^{1/2}\Big\} \le P\Big\{\Big|\sum_{j\le M} T_j\Big| \ge A_1 R(N\log\log N)^{1/2}\Big\}+$$

$$+P\Big\{\Big|\sum_{j\le M} T_j'\Big| \ge A_1 R(N\log\log N)^{1/2}\Big\}+$$

$$+P\Big\{\Big|\sum_{v=C_{M+1}}^{N} \cos 2\pi n_v x\Big| \ge A_1 R(N\log\log N)^{1/2}\Big\} \ll$$

$$\ll \exp(-11R\log\log N)+\exp(-11R\log\log N)+0 \ll \exp(-11R\log\log N).$$

We have used the fact that for all large N, $N-C_{M+1} \le (M+1)^{\varepsilon} < N^{1/2}$ since $\varepsilon<(\varepsilon+1)/2$ by (2.11).

The case where $N\in I_M$ can be proved in the same way. Thus, in general, we have

$$P\Big\{\Big|\sum_{k\le N} \cos 2\pi n_k x\Big| \ge 3A_1 R(N\log\log N)^{1/2}\Big\} \ll \exp(-11R\log\log N).$$

Similarly, we can prove that

$$P\Big\{\Big|\sum_{k\le N} \sin 2\pi n_k x\Big| \ge 3A_1 R(N\log\log N)^{1/2}\Big\} \ll \exp(-11R\log\log N).$$

Hence Proposition 1 is proved.

2.2 Proof of Proposition 2.

Lemma 8. *Put*

$$Z_k = \sum_{v=Q+1}^{Q+k} \psi_v(x).$$

Then we have for any $Q\ge 0$ *and any real number* t

$$P\Big\{\max_{1\le k\le N} |Z_k| > t\Big\} \le 2P\Big\{|Z_n| > t-4\sqrt{N}\Big\}$$

provided that $N\ge N_0$, $Q\le N^{1+\gamma}$ *where* γ *is a positive constant such that* $(1+\gamma)\delta < 1/2$.

The proof of this lemma is exactly the same as the one in [1], pp. 214—216. Note that in the proof the n_k's do not have to be integers.

Lemma 9. *We have for any* $0\le Q\le N^{1+\gamma}$, $N\ge N_0$, *and any real number* $t\ge 3\sqrt{N}$,

$$P\Big\{\max_{1\le k\le N} \Big|\sum_{v=Q+1}^{Q+k} \cos 2\pi n_v x\Big| > 3t\Big\} \le 2P\Big\{\Big|\sum_{v=Q+1}^{Q+N} \cos 2\pi n_v x\Big| > t-2\sqrt{N}\Big\}.$$

Proof. By the definition of ψ_v in (2.14) we have, as before,

$$\|\cos 2\pi n_v x - \psi_v(x)\|_\infty \ll v^{-4}.$$

Thus

$$\max_{1\leq k\leq N}\Big\|\sum_{v=Q+1}^{Q+k} \cos 2\pi n_v x - \psi_v(x)\Big\|_\infty \ll 1.$$

Hence, if t is large enough,

$$P\Big\{\max_{1\leq k\leq N}\Big|\sum_{v=Q+1}^{Q+k} \cos 2\pi n_v x - \psi_v(x)\Big| > t\Big\} = 0.$$

Now Lemma 9 follows from Lemma 8.

Lemma 10. *As* $N\to\infty$,

$$P\Big\{\Big|\sum_{k=N+1}^{2N-1} \cos 2\pi n_k x\Big| \geqq 8A_1 R(N\log\log N)^{1/2}\Big\} \ll \exp(-10R\log\log N).$$

Proof. The probability in question does not exceed the probability

$$P\Big\{\Big|\sum_{k=1}^{2N-1} \cos 2\pi n_k x\Big| \geqq 5A_1 R(N\log\log N)^{1/2}\Big\} +$$

$$+P\Big\{\Big|\sum_{k=1}^{N} \cos 2\pi n_k x\Big| \geqq 3A_1 R(N\log\log N)^{1/2}\Big\} \ll$$

$$\ll P\Big\{\Big|\sum_{k=1}^{2N-1} \cos 2\pi n_k x\Big| \geqq 3A_1 R(2N\log\log 2N)^{1/2}\Big\} + \exp(-10R\log\log N)$$

by Proposition 1.

By Lemmas 9 and 10 we can say that

$$P\Big\{\max_{1\leq m<N}\Big|\sum_{k=N+1}^{N+m} \cos 2\pi n_k x\Big| \geqq 30A_1 R(N\log\log N)^{1/2}\Big\} \leqq$$

$$\leqq 2P\Big\{\Big|\sum_{k=N+1}^{2N-1} \cos 2\pi n_k x\Big| \geqq 10A_1 R(N\log\log N)^{1/2} - 2\sqrt{N}\Big\} \ll$$

$$\ll P\Big\{\Big|\sum_{k=N+1}^{2N-1} \cos 2\pi n_k x\Big| \geqq 8A_1 R(N\log\log N)^{1/2}\Big\} \ll \exp(-10R\log\log N).$$

Similarly, we can show that

$$P\Big\{\max_{1\leq m<N}\Big|\sum_{k=N+1}^{N+m} \sin 2\pi n_k x\Big| \geqq 30A_1 R(N\log\log N)^{1/2}\Big\} \ll \exp(-10R\log\log N).$$

Thus Proposition 2 follows. Also we can choose A and C_1 so large that both propositions will apply.

3. Proof of Theorem 2

We assume that $\alpha \leqq 1$ (the case $\alpha > 1$ is trivial). We choose ε so that

$$\varepsilon > \delta/(1-\delta) \quad \text{and} \quad 1/2+\varepsilon/2(\varepsilon+1) < \alpha. \tag{3.1}$$

This can be done by choosing ε sufficiently close to $\delta/(1-\delta)$ since $1/2+\delta/2<\alpha$ and since $\delta<1$. Put

$$\gamma = 2/(2\alpha+1). \tag{3.2}$$

Thus, by (3.1) $\varepsilon/2(\varepsilon+1)<(1-\gamma)/\gamma\leqq 1/2$. Now choose ε' and β so that

$$\varepsilon/2(\varepsilon+1) < \varepsilon' < (1-\gamma)/\gamma \tag{3.3}$$

and

$$1/2\varepsilon' > \beta > \gamma/2(1-\gamma). \tag{3.4}$$

Put

$$q = \beta/(\beta-1). \tag{3.5}$$

Thus, by (3.4)

$$-q+2q\varepsilon' < -1. \tag{3.6}$$

As in Section 2 we shall prove the following two propositions.

Proposition 3. *Let $\{n_k,\ k\geqq 1\}$ be as in Theorem 2. Then there exist positive constants A and C_1 such that*

$$P\Big\{\Big|\sum_{k=1}^{N}\exp(2\pi i n_k x)\Big| \geqq AR(N\log\log N)^{1/2}\Big\} \leqq$$

$$\leqq C_1\exp(-10R\log\log N)+C_1R^{-1}N^{-1/(\varepsilon+1)}(\log\log N)+$$

$$+C_1R^{-2}N^{-1/2(\varepsilon+1)}(\log\log N)$$

for all $R\geqq 1$, $N\geqq 1$, where ε is defined in (3.1).

Proposition 4. *Let $\{n_k,\ k\geqq 1\}$ be as in Theorem 2. Then*

$$P\Big\{\max_{1\leqq m\leqq N}\Big|\sum_{k=N+1}^{N+m}\exp(2\pi i n_k x)\Big| \geqq AR(N\log\log N)^{1/2}\Big\} \leqq$$

$$\leqq C_1\exp(-10R\log\log N)+C_1R^{-1}N^{-1/(\varepsilon+1)}(\log\log N)+$$

$$+C_1R^{-2}N^{-1/2(\varepsilon+1)}(\log\log N)$$

for all $R\geqq 1$, $N\geqq 1$. Here A and C_1 are the same as in Proposition 3.

The proofs of these two propositions will be given in Section 3.1 and 3.2.

To apply the propositions we need to define some events. For integers h, $n \geqq 0$, let

$$G(n, h) = \{F(0, 2^n, h) \geqq A |h|^{2\varepsilon'} \varphi(2^n)\};$$

$$G_n = \bigcup_{1 \leqq |h| \leqq 2^n} G(n, h);$$

$$H(n, h) = \{\max_{1 \leqq m < 2^n} F(2^n, m, h) \geqq A |h|^{2\varepsilon'} \varphi(2^n)\};$$

$$H_n = \bigcup_{1 \leqq |h| \leqq 2^n} H(n, h)$$

by using definitions in (2.1), (3.4) and (2.3). Here A is the constant appearing in Proposition 3.

Taking Propositions 3 and 4 for granted we see that

$$P(G_n) \ll \sum_{1 \leqq |h| \leqq 2^n} \exp(-2|h|^{2\varepsilon'} \log n) + (|h|^{-2\varepsilon'} 2^{(-1/(\varepsilon+1))n} + |h|^{-4\varepsilon'} 2^{(-1/2(\varepsilon+1))n}) \log n) \ll$$

$$\ll n^{-2} + 2^{(1-2\varepsilon'-1/(\varepsilon+1))n} \log n + 2^{(1-4\varepsilon'-1/2(\varepsilon+1))n} \log n;$$

$$P(H_n) \ll n^{-2} + 2^{(1-2\varepsilon'-1/(\varepsilon+1))n} \log n + 2^{(1-4\varepsilon'-1/2(\varepsilon+1))n} \log n.$$

Since $1-2\varepsilon' < 1/(\varepsilon+1)$ and $1-4\varepsilon' < 1/2(\varepsilon+1)$ by (3.3) and (3.1), we have

$$\sum_{n \geqq 0} P(G_n) < \infty, \quad \sum_{n \geqq 0} P(H_n) < \infty.$$

Thus by Borel—Cantelli Lemma,

$$F(0, N, h) \ll |h|^{2\varepsilon'} \varphi(N) \quad \text{a.s.}$$

for all $1 \leqq |h| \leqq N/2$.

Now for each $f \in A_\alpha$ $(\alpha > 1/2 + \delta/2)$ we have as in (2.10)

$$(3.7) \quad \left|\sum_{k \leqq N} f(n_k x)\right| \leqq \Big(\sum_{1 \leqq |h| \leqq N/2} |a_h|^\beta |h|^\beta\Big)^{1/\beta} \Big(\sum_{1 \leqq |h| \leqq N/2} |h|^{-q} F^q(0, N, h)\Big)^{1/q} + N^{1/2} \ll$$

$$\ll 1\Big(\sum_{|h| \geqq 1} |h|^{-q+2\varepsilon' q}\Big)^{1/q} \varphi(N) + N^{1/2} \ll \varphi(N)$$

by (3.4), (3.5), (3.6) and (2.6).

We have used the fact that, by (3.4) and (3.2), $\beta > \gamma$. Thus the argument in Section 6.32 on page 137 of [15] implies

$$(3.8) \quad \sum_{h=2^{v-1}+1}^{2^v} |a_h|^\beta |h|^\beta \ll 2^{\beta v} 2^{v((1/2)-(\beta/\gamma))} \ll 2^{v(\beta+(1/2)-(\beta/\gamma))}.$$

Since $\beta + (1/2) - (\beta/\gamma) < 0$ by (3.4), (3.8) implies

$$\sum_{|h| \geqq 1} |a_h|^\beta |h|^\beta \ll 1.$$

3.1 **Proof of Proposition 3.** The proof of Proposition 3 is similar to that of Proposition 1. We begin with an estimate for fourth moments.

Lemma 11. *For integers* $M \geqq 0$, $N \geqq 1$, *put* $S_{M,N} = \sum_{j=M+1}^{M+N} \cos 2\pi n_j x$. *Thus*

$$\int_0^1 S_{M,N}^4 \ll N^2$$

where the constant implied by $\ll$ *depends on bound in* B_2*-condition of the sequence* $\{n_k\}$.

This lemma is Lemma (5.4) of [3]. The proof is quite simple. It is strongly based on the assumptions that $n_k \in \mathbf{Z}$ and satisfies B_2-condition.

Lemma 12. *For* $1 \leqq j < k$ *we have* $n_j/n_k \leqq 2^{-c(k-j)/k^\delta}$.

Proof.

$$n_k/n_j \geqq \prod_{i=j}^{k-1} (1+c/i^\delta) > (1+c/k^\delta)^{k-j} > 2^{c(k-j)/k^\delta}$$

by (1.2).

Divide $\mathbf{Z}^+$ into blocks (without gaps) such that

$$H_1 < I_1 < H_2 < I_2 < \ldots < H_j < I_j < \ldots \quad \text{(say)},$$

$$\operatorname{card}(H_j) = [j^\varepsilon] = \operatorname{card}(I_j)$$

where ε is defined in (3.1). We define c_j, d_j, m_v, ϱ_v, ψ_v, T_j, D_j, and $\bar{D}_j$ as in Section 2.1.

The proofs of the following three lemmas are the same as the proofs of Lemma 2, 3 and 4 respectively.

Lemma 13. $\|T_j - D_j\|_\infty \ll C_j^{-3}$.

Lemma 14. $E(T_j|D_1, \ldots, D_{j-1}) \ll j^{-2}$ *a.s.*

Lemma 15. $E(T_j^2|D_1, \ldots, D_{j-1}) \ll j^\varepsilon$ *a.s.*

Lemma 16. (1) $E(D_j|D_1, \ldots, D_{j-1}) \ll j^{-2}$ *a.s.*
(2) $E(D_j^2|D_1, \ldots, D_{j-1}) \ll j^\varepsilon$ *a.s.*
(3) $E(\bar{D}_j^2|D_1, \ldots, D_{j-1}) \ll j^\varepsilon$ *a.s.*

Lemma 16 can be proved as in Lemma 5.

Put

(3.9) $$t_j = j^{(\varepsilon+1)/2}/4\sqrt{\log\log j^{\varepsilon+1}},$$

and define random variables

(3.10) $$D_j^* = D_j I[|D_j| \leqq t_M] \qquad (j \leqq M),$$

$$\bar{D}_j^* = D_j^* - E(D_j^*|D_1, \ldots, D_{j-1}) \quad (j \leqq M).$$

Thus

(3.11) $$E(\bar{D}_j^{*2}|D_1, \ldots, D_{j-1}) \ll j^\varepsilon \quad \text{a.s.}$$

since

$$E(\bar{D}_j^{*2}|D_1, \ldots, D_{j-1}) \leqq E(D_j^{*2}|D_1, \ldots, D_{j-1}) \leqq E(D_j^2|D_1, \ldots, D_{j-1}) \ll j^\varepsilon \quad \text{a.s.}$$

by Lemma 16(2).

Lemma 17. *Let B ($\geqq 2$) be the constant implied by $\ll$ in* (3.11). *Then we have as $M \to \infty$*

$$P\{|\sum_{j \leqq M} \bar{D}_j^*| \geqq 8RB(M^{\varepsilon+1}\log\log M^{\varepsilon+1})^{1/2}\} \ll \exp(-12RB\log\log M^{\varepsilon+1}).$$

Proof. Put

$$u_n = \sum_{j \leqq n} \bar{D}_j^*, \quad n \leqq M$$
$$= u_M, \qquad n > M;$$
$$s_n^2 = \sum_{j \leqq n} E(\bar{D}_j^{*2}|D_1, \ldots, D_{j-1}), \quad n \leqq M$$
$$= s_M^2, \qquad n > M;$$
$$c = 2t_M, \quad \lambda = 2(\log\log M^{\varepsilon+1})^{1/2} M^{-(\varepsilon+1)/2}, \quad K = 4RBM^{\varepsilon+1},$$
$$T_n = \exp\left(\lambda u_n - (1/2)\lambda^2(1+(1/2)\lambda c)s_n^2\right).$$

Thus $\{u_n, n \geqq 1\}$ is a martingale. Moreover, $\bar{D}_j^* \leqq c$ a.s. and $\lambda c \leqq 1$. Now the proof can be finished as in the proof of Lemma 6.

Lemma 18.

$$P\{|\sum_{j \leqq M} E(D_j^*|D_1, \ldots, D_{j-1})| \geqq 2RB(M^{\varepsilon+1}\log\log M^{\varepsilon+1})^{1/2}\} \ll R^{-1}M^{-1}(\log\log M).$$

Proof. The probability in question is, by (3.10),

$$\begin{aligned}
&\leqq P\{|\sum_{j \leqq M} E(D_j|D_1, \ldots, D_{j-1})| \geqq RB(M^{\varepsilon+1}\log\log M^{\varepsilon+1})^{1/2}\} + \\
&+ P\{\sum_{j \leqq M} E(D_j I[|D_j| > t_M]|D_1, \ldots, D_{j-1}) \geqq RB(M^{\varepsilon+1}\log\log M^{\varepsilon+1})^{1/2}\} \leqq \\
&\leqq 0 + \sum_{j \leqq M} E|E(D_j I[|D_j| > t_M]|D_1, \ldots, D_{j-1})|/RM^{(\varepsilon+1)/2}(\log\log M)^{1/2} \leqq \\
&\leqq \sum_{j \leqq M} E^{1/4}(D_j^4)P^{3/4}\{|D_j| > t_M\}/RM^{(\varepsilon+1)/2}(\log\log M)^{1/2} \ll \\
&\ll \sum_{j \leqq M} E^{1/2}(T_j^4)E^{3/4}(D_j^4)t_M^{-3}/RM^{(\varepsilon+1)/2}(\log\log M)^{1/2} \ll \\
&\ll \sum_{j \leqq M} j^{\varepsilon/2} j^{3\varepsilon/2} M^{-3(\varepsilon+1)/2}(\log\log M)^{3/2}/RM^{(\varepsilon+1)/2}(\log\log M)^{1/2} \ll \\
&\ll R^{-1}M^{2\varepsilon+1-2(\varepsilon+1)}(\log\log M) = R^{-1}M^{-1}(\log\log M)
\end{aligned}$$

by Lemma 16(1), Hölder's inequality, Lemma 13, Markov inequality, and Lemma 11.

Lemma 19.

$$P\{|\sum_{j\leqq M}\bar{D}_j|\geqq 12RB(M^{\varepsilon+1}\log\log M^{\varepsilon+1})^{1/2}\}\ll$$
$$\ll\exp(-12RB\log\log M)+R^{-1}M^{-1}(\log\log M)+R^{-2}M^{-1/2}(\log\log M).$$

Proof. Put $\lambda=10RB(M^{\varepsilon+1}\log\log M^{\varepsilon+1})^{1/2}$. Now the probability in question is

$$\leqq P\{|\sum_{j<M}D_j|>\lambda\}=$$
$$=P\{|\sum_{j\leqq M}D_j|>\lambda;\ \max_{j\leqq M}|D_j|\leqq t_M\}+P\{|\sum_{j\leqq M}D_j|>\lambda;\ \max_{j\leqq M}|D_j|>t_M\}\leqq$$
$$\leqq P\{|\sum_{j\leqq M}D_j^*|>\lambda\}+P^{1/2}\{|\sum_{j\leqq M}D_j|>\lambda\}P^{1/2}\{\max_{j\leqq M}|D_j|>t_M\}\leqq$$
$$\leqq P\{|\sum_{j\leqq M}\bar{D}_j^*|>8RB(M^{\varepsilon+1}\log\log M^{\varepsilon+1})^{1/2}\}+$$
$$+P\{|\sum_{j\leqq M}E(D_j^*|D_1,\dots,D_{j-1})|>2RB(M^{\varepsilon+1}\log\log M^{\varepsilon+1})^{1/2}\}+$$
$$+P^{1/2}\left\{|\sum_{j\leqq M}T_j|>\frac{\lambda}{2}\right\}\Big(\sum_{j\leqq M}P\{|D_j|>t_M\}\Big)^{1/2}\ll$$
$$\ll\exp(-12RB\log\log M)+R^{-1}M^{-1}(\log\log M)+$$
$$+(M^{2(\varepsilon+1)}/R^4M^{2(\varepsilon+1)})^{1/2}\Big(t_M^{-4}\sum_{j\leqq M}j^{2\varepsilon}\Big)^{1/2}\ll$$
$$\ll\exp(-12RB\log\log M)+R^{-1}M^{-1}(\log\log M)+R^{-2}M^{-1/2}(\log\log M)$$

by (3.9), (3.10), Lemma 17, Lemma 18, Hölder's inequality, and Markov inequality.

Lemma 20. *There is a positive constant A_1 ($\geqq 1$) such that*

$$P\{|\sum_{j\leqq M}T_j|\geqq A_1R(M^{\varepsilon+1}\log\log M^{\varepsilon+1})^{1/2}\}\ll$$
$$\ll\exp(-12R\log\log M^{\varepsilon+1})+(R^{-1}M^{-1}+R^{-2}M^{-1/2})\log\log M.$$

The proof is very simple (see Lemma 7). Similarly,

$$P\{|\sum_{j\leqq M}T_j'|\geqq A_1R(M^{\varepsilon+1}\log\log M^{\varepsilon+1})^{1/2}\}\ll$$
$$\ll\exp(-12R\log\log M^{\varepsilon+1})+(R^{-1}M^{-1}+R^{-2}M^{-1/2})\log\log M$$

where

$$T_j'=\sum_{v\in I_j}\cos 2\pi n_v x.$$

Lemma 21.

$$P\{|\sum_{k\leqq N}\cos 2\pi n_k x|\geqq 3A_1R(N\log\log N)^{1/2}\}\ll$$
$$\ll\exp(-11R\log\log N)+(R^{-1}N^{-1/(\varepsilon+1)}+R^{-2}N^{-1/2(\varepsilon+1)})\log\log N.$$

Proof. Assume first that $N \in H_{M+1}$ for some M. Then the lemma follows from the following estimation:

$$P\{|\sum_{v=C_{M+1}}^{N} \cos 2\pi n_v x| \geqq AR(N \log\log N)^{1/2}\} \ll M^{2\varepsilon}/R^4 N^2 \ll R^{-4} N^{-2/(1+\varepsilon)}$$

by Markov inequality and Lemma 11.

For the case $N \in I_{M+1}$, the proof is the same as above.

Similar to Lemma 21, one can prove that

$$P\{|\sum_{k \leqq N} \sin 2\pi n_k x| \geqq 3A_1 R(N \log\log N)^{1/2}\} \ll \tag{3.12}$$

$$\ll \exp(-11R \log\log N) + (R^{-1} N^{-1/(\varepsilon+1)} + R^{-2} N^{-1/2(\varepsilon+1)}) \log\log N.$$

It is clear now that Proposition 3 follows from Lemma 21 and (3.12).

3.2 Proof of Proposition 4.

Theorem A (Theorem 12.2 of [5]). *Let $\xi_1, \ldots, \xi_m$ be random variables. Let $s_k = \xi_1 + \ldots + \xi_k$ ($s_0 = 0$), and put $M_m = \max_{0 \leqq k \leqq m} |s_k|$. Suppose for some $\gamma \geqq 0$, $\alpha > 1$, and some $u_1, \ldots, u_m > 0$,*

$$P\{|s_j - s_i| \geqq \lambda\} \leqq (1/\lambda^\gamma)(\sum_{i<\varrho\leqq j} u_\varrho)^\alpha, \quad 0 \leqq i < j \leqq m$$

for all $\lambda > 0$. Then for all $\lambda > 0$,

$$P\{M_m > \lambda\} \leqq (k/\lambda^\gamma)(\sum_{0<\varrho\leqq m} u_\varrho)^\alpha$$

where K is a constant depending only on γ and α.

Lemma 22. *For $k \geqq 1$, put $Z_k = \sum_{v=N+1}^{N+k} \cos 2\pi n_v x$. Then, for some $c_0 \geqq 2$, we have*

$$P\{\max_{1\leqq k\leqq N} |Z_k| \geqq 24 A_1 R C_0 (N \log\log N)^{1/2}\} \ll$$

$$\ll \exp(-11R \log\log N) + (R^{-1} N^{-1/(\varepsilon+1)} + R^{-2} N^{-1/2(\varepsilon+1)}) \log\log N$$

where A_1 is the constant appearing in Lemma 20.

Note that under the condition (1.2) of $\{n_k\}$, some parts of the proof of Lemma 8 (or Lemma (6) of [1]) need to be changed.

Proof of Lemma 22. Suppose that $N+1 \in H_m$ and $2N \in H_M$ for some m and M. (The proofs for other cases are the same.)

Let t be any number $\geqq 3\sqrt{N}$. Thus

$$P\Big\{\max_{1\leqq k\leqq N}|Z_k|>6t\Big\}\leqq P\Big\{\max_{m\leqq k\leqq M}\Big|\sum_{j=m}^{k}T_j\Big|>2t\Big\}+$$

$$+P\Big\{\max_{m\leqq k\leqq M}\max_{j\in H_k}\Big|\sum_{v=c_k}^{j}\cos 2\pi n_v x\Big|>t\Big\}+P\Big\{\max_{m\leqq k\leqq M-1}\Big|\sum_{j=m}^{k}T_j'\Big|>2t\Big\}+$$

$$+P\Big\{\max_{m\leqq k\leqq M-1}\max_{j\in I_k}\Big|\sum_{v=d_k+1}^{j}\cos 2\pi n_v x\Big|>t\Big\}=I_1+I_1'+I_2+I_2'\quad\text{(say)}.$$

By Theorem A, Lemma 11 together with Markov inequality, we have

$$I_1'\leqq\sum_{k=m}^{M}P\Big\{\max_{j\in H_k}\Big|\sum_{v=c_k}^{j}\cos 2\pi n_v x\Big|>t\Big\}\ll$$

$$\ll\sum_{k=m}^{M}k^{2\varepsilon}/t^4\ll M^{2\varepsilon+1}/t^4\ll N^{(2\varepsilon+1)/(\varepsilon+1)}/t^4.$$

Similarly, $I_2'\ll N^{(2\varepsilon+1)/(\varepsilon+1)}/t^4$. By Lemma 16(1),

$$I_1\leqq P\Big\{\max_{m\leqq k\leqq M}\Big|\sum_{j=m}^{k}\bar D_j\Big|>t\Big\}+P\Big\{\max_{m\leqq k\leqq M}\Big|\sum_{j=m}^{k}(T_j-\bar D_j)\Big|>t\Big\}\leqq \tag{3.13}$$

$$\leqq P\Big\{\max_{m\leqq k\leqq M}\Big|\sum_{j=m}^{k}\bar D_j\Big|>t\Big\}.$$

Write $\bar Z_k=\sum_{j=m}^{k}\bar D_j\ (m\leqq k\leqq M)$, and put

$$A=\Big\{\max_{m\leqq k\leqq M}\bar Z_k>t\Big\},\quad A_m=\{\bar Z_m>t\},$$

$$A_k=\{\bar Z_m\leqq t,\ \bar Z_{m+1}\leqq t,\ \dots,\ \bar Z_{k-1}\leqq t,\ \bar Z_k>t\}\quad(m<k\leqq M),$$

$$B_k=\big\{\bar Z_m-\bar Z_k>-C_0\sqrt{N}\big\}\quad(m\leqq k\leqq M),\quad C=\big\{\bar Z_M>t-C_0\sqrt{N}\big\}.$$

Thus A_kB_k are pairwise disjoint. Also $\bigcup_{k=m}^{M}A_kB_k\subset C$.

On B_k^c we have $(\bar Z_M-\bar Z_k)^2>C_0^2N$. Thus

$$P(A_kB_k^c)=\int_{A_k}1_{B_k^c}\leqq C_0^{-2}N^{-1}\int_{A_k}(\bar Z_M-\bar Z_k)^2= \tag{3.14}$$

$$=C_0^{-2}N^{-1}\sum_{\substack{I\subset A_k\cap\sigma(D_1,\dots,D_k)\\ P(I)=2^{-md_k}}}\int_I(\bar Z_M-\bar Z_k)^2.$$

For each such I in the summation,

(3.15)

$$\int_I (\bar{Z}_M-\bar{Z}_k)^2 = \int_I E((\bar{Z}_M-\bar{Z}_k)^2|D_1, \dots, D_k) = \int_I E((\sum_{j=k+1}^{M} \bar{D}_j)^2|D_1, \dots, D_k) \leqq$$

$$\leqq \|E((\sum_{j=k+1}^{M} \bar{D}_j)^2|D_1, \dots, D_k)\|_\infty P(I) = \|E(\sum_{j=k+1}^{M} \bar{D}_j^2|D_1, \dots, D_k)\|_\infty P(I) \leqq$$

$$\leqq (\sum_{j=k+1}^{M} j^s)P(I) \leqq C_0 NP(I) \quad \text{for some } C_0.$$

Hence (3.14) and (3.15) imply

$$P(A_k B_k^c) \leqq C_0^{-2} N^{-1}(C_0 N) \sum_I P(I) = C_0^{-1} P(A_k) \leqq (1/2) P(A_k).$$

Now

$$(1/2)P(A) = (1/2) \sum_{k=m}^{M} P(A_k) \leqq \sum_{k=m}^{M} (P(A_k)-P(A_k B_k^c)) = \sum_{k=m}^{M} P(A_k B_k) \leqq P(C).$$

This proves that

$$P\{\max_{m\leqq k\leqq M} \sum_{j=m}^{M} \bar{D}_j > t\} \leqq 2P\{\sum_{j=m}^{M} \bar{D}_j > t - C_0\sqrt{N}\}.$$

Similarly,

$$P\{\max_{m\leqq k\leqq M} -\sum_{j=m}^{k} \bar{D}_j > t\} \leqq 2P\{-\sum_{j=m}^{M} \bar{D}_j > t - C_0\sqrt{N}\}.$$

Hence

$$P\{\max_{m\leqq k\leqq M} |\sum_{j=m}^{k} \bar{D}_j| > t\} \leqq 2P\{|\sum_{j=m}^{M} \bar{D}_j| > t - C_0\sqrt{N}\}.$$

Therefore, by (3.13)

$$I_1 \leqq 2P\{|\sum_{j=m}^{M} \bar{D}_j| > t - C_0\sqrt{N}\}.$$

But

$$P\{|\sum_{j=m}^{M} \bar{D}_j| > t - C_0\sqrt{N}\} \leqq P\{|\sum_{j=m}^{M} T_j| > t/2 - (C_0/2)\sqrt{N}\}.$$

Hence

$$I_1 \ll P\{|\sum_{j=m}^{M} T_j| > t/2 - (C_0/2)\sqrt{N}\}.$$

Similarly,

$$I_2 \ll P\{|\sum_{j=m}^{M-1} T_j| > t/2 - (C_0/2)\sqrt{N}\}.$$

We can conclude now that if $t=4ARC_0(N\log\log N)^{1/2}$, then

$$P\Big\{\max_{m\le k\le M}|Z_k|>6t\Big\}\ll$$

$$\ll P\Big\{\Big|\sum_{j=m}^{M}T_j\Big|>t/2-(C_0/2)\sqrt{N}\Big\}+P\Big\{\Big|\sum_{j=m}^{M-1}T_j'\Big|>t/2-(C_0/2)\sqrt{N}\Big\}+N^{(2\varepsilon+1)/(\varepsilon+1)}/t^4\ll$$

$$\ll P\Big\{\Big|\sum_{j=1}^{M}T_j\Big|>t/4-(C_0/4)\sqrt{N}\Big\}+P\Big\{\Big|\sum_{j=1}^{m-1}T_j\Big|>t/4-(C_0/4)\sqrt{N}\Big\}+$$

$$+P\Big\{\Big|\sum_{j=1}^{M-1}T_j'\Big|>t/4-(C_0/4)\sqrt{N}\Big\}+P\Big\{\Big|\sum_{j=1}^{m-1}T_j'\Big|>t/4-(C_0/4)\sqrt{N}\Big\}+N^{(2\varepsilon+1)/(\varepsilon+1)}/t^4\ll$$

$$\ll\exp(-11R\log\log N)+(R^{-1}N^{1/(\varepsilon+1)}+R^{-2}N^{-1/2(\varepsilon+1)})\log\log N+N^{(2\varepsilon+1)/(\varepsilon+1)}t^{-4}\ll$$

$$\ll\exp(-11R\log\log N)+(R^{-1}N^{1/(\varepsilon+1)}+R^{-2}N^{-1/2(\varepsilon+1)})\log\log N.$$

Putting $A>24A_1C_0$, Proposition 4 is proved. We choose A large enough that both Proposition 3 and 4 will apply.

4. Further results

Recently, I found an improvement of Theorem 2. The proof requires only little work beyond the one of Theorem 2. We shall give a sketch of the proof.

Theorem 3. *Suppose that the sequence $\{n_k\}$ of integers satisfies (1.2) and furthermore, for any $v>0$, the number of solutions of $n_k\pm n_l=v$ $(1\le k, l\le N)$ is at most BN^η with constants $B>0$, $\eta<(1-\delta)/2$. Then for each α with $\alpha>1/2+\delta/2+\eta$,*

$$\limsup_{N\to\infty}\sup_{f\in A_\alpha}\Big|\sum_{k=1}^{N}f(n_kx)\Big|/(N\log\log N)^{1/2}\ll 1$$

for almost all $x\in[0,1]$, where the constant implied by $\ll$ depends on δ, c, B and α.

The case $\eta=0$ in the above theorem means that the sequence $\{n_k\}$ satisfies condition B_2; in this special case we have Theorem 2.

4.1 Proof of Theorem 3. We assume that $\alpha\le 1$. We choose ε so that

$$\varepsilon>\delta/(1-\delta)\quad\text{and}\quad 1/2+\varepsilon/2(\varepsilon+1)+\eta<\alpha. \tag{4.1}$$

(This can be done by choosing ε sufficiently close to $\delta/(1-\delta)$ since $1/2+\delta/2+\eta<\alpha$ and $\delta<1$.) Put

$$\gamma=2/(2\alpha+1). \tag{4.2}$$

Thus, by (4.1),

(4.3) $$\varepsilon/2(\varepsilon+1) < (1-\gamma)/\gamma \leqq 1/2.$$

Now choose ε' so that

(4.4) $$\max\left(\frac{\varepsilon}{2(\varepsilon+1)}, \frac{2\varepsilon+1}{8(\varepsilon+1)}+\frac{\eta}{2}, \frac{1}{2}-\frac{1}{2(\varepsilon+1)}+\eta\right) < \varepsilon' <$$
$$< \min\left(\frac{1-\gamma}{\gamma}, 1-\frac{\delta}{2}-\frac{1}{2(\varepsilon+1)}\right).$$

(It is easy to verify that max<min in (4.4).) We choose β so that

(4.5) $$1/2\varepsilon' > \beta > \gamma/2(1-\gamma)$$

and put

(4.6) $$q = \beta/(\beta-1).$$

Thus, by (4.4) and (4.5), we have

(4.7) $$-q+2q\varepsilon' < -1.$$

We also have

(4.8) $$1-2\varepsilon' < (1-2\eta(\varepsilon+1))/(\varepsilon+1)$$

and

$$1-4\varepsilon' < 1-(2\varepsilon+1)/2(\varepsilon+1)-2\eta$$

by (4.4).

We define the events, for integers h and $n\geqq 0$,

$$G(n, h) = \{F(0, 2^n, h) \geqq A|h|^{2\varepsilon'}\varphi(2^n)\};$$
$$G_n = \bigcup_{1\leqq |h|\leqq 2^n} G(n, h);$$
$$H(n, h) = \{\bigcup_{1\leqq m<2^n} F(2^n, m, h) \geqq A|h|^{2\varepsilon'}\varphi(2^n)\};$$
$$H_n = \bigcup_{1\leqq |h|\leqq 2^n} H(n, h)$$

where A is the constant appearing in Proposition 5 below.

Proposition 5. *Let $\{n_k\}$ be as in Theorem 3. Then there exist positive constants A and C such that*

$$P\{|\sum_{k=1}^{N} \exp(2\pi i n_k x)| \geqq AR(N\log\log N)^{1/2}\} \leqq$$
$$\leqq C[\exp(-10R\log\log N)+R^{-1}N^{-(1-2\eta(\varepsilon+1))/(\varepsilon+1)}\log\log N+$$
$$+R^{-2}N^{-(1-(2\varepsilon+1)/2(\varepsilon+1)-2\eta)}\log\log N]$$

for all $R\geqq 1$, $N\geqq 1$.

Proposition 6. *Let $\{n_k\}$ be as in Theorem 3. Then*

$$P\{\max_{1\leqq m<N}\Big|\sum_{k=N+1}^{N+m}\exp(2\pi i n_k x)\Big|\geqq AR(N\log\log N)^{1/2}\}\leqq$$
$$\leqq C\,[\exp(-10R\log\log N)+R^{-1}N^{-(1-2\eta(\varepsilon+1))/(\varepsilon+1)}\log\log N+$$
$$+R^{-2}N^{-(1-(2\varepsilon+1)/2(\varepsilon+1)-2\eta)}\log\log N]$$

for all $R\geqq 1$, $N\geqq 1$.

Here A and C are the same constants as in Proposition 5.

Taking Proposition 5 and 6 for granted we see that

$$P(G_n)\ll\sum_{1\leqq|h|\leqq 2^n}[\exp(-2|h|^{2\varepsilon'}\log n)+$$
$$+|h|^{-2\varepsilon'}2^{-(1-2\eta(\varepsilon+1))n/(\varepsilon+1)}\log n+|h|^{-4\varepsilon'}2^{-(1-(2\varepsilon+1)/2(\varepsilon+1)-2\eta)n}\log n]\ll$$
$$\ll n^{-2}+2^{(1-2\varepsilon'-(1-2\eta(\varepsilon+1))/(\varepsilon+1))n}\log n+2^{[1-4\varepsilon'-(1-(2\varepsilon+1)/2(\varepsilon+1)-2\eta)]n}\log n;$$
$$P(H_n)\ll n^{-2}+2^{(1-2\varepsilon'-(1-2\eta(\varepsilon+1))/(\varepsilon+1))n}\log n+2^{[1-4\varepsilon'-(1-(2\varepsilon+1)/2(\varepsilon+1)-2\eta)]n}\log n.$$

Since $1-2\varepsilon'<(1-2\eta(\varepsilon+1))/(\varepsilon+1)$ and $1-4\varepsilon'<1-(2\varepsilon+1)/2(\varepsilon+1)-2\eta$ by (4.8), we obtain

$$\sum_{n\geqq 0}P(G_n)<\infty,\quad \sum_{h\geqq 0}P(H_n)<\infty.$$

Thus, by Borel—Cantelli Lemma,

$$F(0,N,h)\ll|h|^{2\varepsilon'}\varphi(N)\quad\text{a.s.}$$

for all $1\leqq|h|\leqq N/2$. Now (3.7) can be obtained using ε', β and q in (4.4), (4.5) and (4.6) respectively.

Since the proofs of Proposition 5 and 6 are similar to those of Proposition 3 and 4 of Section 3, we omit them.

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